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Axially Symmetric Bianchi Type-IX Inflationary Universe in General Relativity

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Abstract Axially symmetric Bianchi type-IX space-time in the presence of mass less scalar field with a flat potential V is investigated. To get an inflationary universe, we have considered a flat region in which potential V is constant. Some physical and kinematical properties of the model are discussed.

Keywords Axially symmetric Bianchi type-IX space-time · Inflationary universe · General relativity

1 Introduction

Bianchi type cosmological models are important in the sense that these are homogeneous and anisotropic from which the process of isotropization of the universe is studied through the passage of time. Moreover, from the theoretical point of view anisotropic universe have a greater generality than isotropic models. Many relativists have taken interest in studying Bianchi type-IX universe because familiar solutions like Robertson Walker universe with positive curvature, the de-sitter universe, the Taub-NUT solutions etc are of Bianchi type IX space time. Actually the study of anisotropic models was started after the discovery of microwave background radiation in 1965. Chakraborty [5] have investigated a class of cosmological solutions of massive strings in Bianchi type-IX space-time using a supplementary condition $a = \alpha b^n$ between metric potentials a and b, where α and n are constants. Bali and Dave [1], Bali and Yadav [3] have investigated Bianchi type IX string as well as viscous fluid models in general relativity. Rahaman et al. [9] have investigated Bianchi type-IX string cosmological model in Lyra geometry. Also Reddy et al. [11] have investigated Bianchi type-IX string cosmological model in a scalar tensor theory of gravitation.

In recent years there has been a lot of interest in cosmological models of the universe which are important in understanding the mysteries of the early stages of it's evolution. In particular, inflationary models of the universe play a vital role in solving a number of

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outstanding problems in cosmology like the homogeneity, the isotropy and flatness of the observed universe. That is, it has undergone at an early stage a period of exponential expansion known as inflation. Guth [6], Linde [8] and La and Steinhardt [7] are some of the authors who have investigated several aspects of inflationary universes in general relativity.

Scalar fields are the simplest classical fields and there exists an extensive literature containing numerous solutions of the Einstein equations where the scalar field is minimally coupled to the gravitational field. In particular, self-interacting scalar fields play a central role in the study of inflationary cosmology. Bhattacharjee and Baruah [4], Bali and Jain [2], Reddy and Naidu [11] have studied the role of self-interacting scalar fields in inflationary cosmology. Very recently Reddy et al. [12] have discussed a plane symmetric Bianchi type-I inflationary universe in general relativity.

In this paper, we have investigated axially symmetric Bianchi type-IX inflationary cosmological model in the presence of mass less scalar field with a flat potential in general relativity. To get a determinate solution, we have considered a flat region in which potential is constant. We have also assumed the relation between metric coefficients for this purpose.

2 Field Equations and Its Solutions

We consider the axially symmetric Bianchi type-IX metric in the form

$$ds^2 = -dt^2 + a^2dx^2 + b^2dy^2 + (b^2 \sin^2 y + a^2 \cos^2 y)dz^2 - 2a^2 \cos y dx dz, \tag{1}$$

where a and b are functions of 't' alone.

In the case of gravity minimally coupled to a scalar field $V(\phi)$, the Lagrangian is

$$L = \int \left[R - \frac{1}{2} g^{ij} \phi_{,i} \phi_{,j} - V(\phi) \right] \sqrt{-g} dx^4 \tag{2}$$

which on variation of L with respect to dynamical fields leads to Einstein's field equations

$$G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R = -T_{ij} \tag{3}$$

with

$$T_{ij} = \phi_{,i} \phi_{,j} - \left[\frac{1}{2} \phi_{,k} \phi^{,k} + V(\phi) \right] g_{ij} \tag{4}$$

$$\phi^{i}_{;i} = -\frac{dV}{d\phi}, \tag{5}$$

where comma and semicolon indicates ordinary and covariant differentiation respectively.

Other symbols have their usual meaning and units are taken so that

$$8\pi G = C = 1.$$

Now the Einstein's field equations (3) for the metric (1) are given by

$$2\frac{b_{44}}{b} + \frac{(b_4)^2}{b^2} + \frac{1}{b^2} - \frac{3}{4}\frac{a^2}{b^4} = \frac{1}{2}\left(\phi_4^2 + V(\phi)\right) \tag{6}$$

$$\frac{a_{44}}{a}+\frac{b_{44}}{b}+\frac{a_4b_4}{ab}+\frac{1}{4}\frac{a^2}{b^4}=\frac{1}{2}\left(\phi_4^2+V(\phi)\right) \tag{7}$$

$$2\frac{a_4b_4}{ab}+\frac{(b_4)^2}{b^2}-\frac{1}{4}\frac{a^2}{b^4}+\frac{1}{b^2}=-\frac{1}{2}\left(\phi_4^2-V(\phi)\right) \tag{8}$$

and (5) for the scalar field takes the form

$$\left(\frac{a_4}{a}+2\frac{b_4}{b}\right)\phi_4+\phi_{44}=\frac{dV}{d\phi}. \tag{9}$$

Here the subscript 4 denotes differentiation with respect to t.

Stein-Schabes [10] have shown that Higgs field ϕ with potential $V(\phi)$ has a flat region and the field evolves slowly but the universe expands in an exponential way due to vacuum field energy. It is assumed that the scalar field will take sufficient time to cross the flat region of the potential so that the universe expands sufficiently to become homogeneous and isotropic on the scale of the order of the horizon size. The flat part of the potential is naturally associated with the vacuum energy that dominates the dynamics for a period of time and identify this vacuum energy with an effective cosmological constant. Therefore, we have assumed the flat region where the potential is constant,

$$\text{i.e.}\quad V(\phi)=\text{const.}$$

From the field equations (6) and (7), we obtain

$$\frac{b_{44}}{b}-\frac{a_{44}}{a}+\frac{(b_4)^2}{b^2}-\frac{a_4b_4}{ab}-\frac{a^2}{b^4}+\frac{1}{b^2}=0. \tag{10}$$

For complete determinacy of the system, one extra condition is needed, therefore, we assume a relation between metric coefficients given by

$$a=b^n,\quad n\neq 1 \tag{11}$$

where n is an arbitrary constant.

With the help of (11), (10) reduces to

$$\frac{b_{44}}{b}+(n+1)\frac{b_4^2}{b^2}=\frac{1}{(n-1)b^2}-\frac{1}{n-1}b^{2n-4}. \tag{12}$$

Solving (13), we obtain

$$(b_4)^2=\frac{1}{n^2-1}-\frac{1}{2n^2-2n}b^{2n-2}+Db^{-2n-2}, \tag{13}$$

where D is constant of integration.

Equation (13) is the first order differential equation, this equation can be written in an integral form as

$$\int\left[\frac{1}{n^2-1}-\frac{1}{2n^2-2n}b^{2n-2}+Db^{-2n-2}\right]^{-\frac{1}{2}}db=\pm(t-t_0), \tag{14}$$

where t_0 is constant of integration.

Equation (14) can be solved for arbitrary values of n and D.

Now, for different choices of arbitrary values of n and D, we will examine the following cases.

Case I:
Let us consider $D=0$ and $n=\frac{1}{2}$ in (14), we obtain

$$-\sqrt{b\left(\frac{3}{2}-b\right)}+\frac{3}{4}\sin^{-1}\left(\frac{4b-3}{3}\right)=\pm\frac{4}{\sqrt{6}}(t-t_0). \tag{15}$$

From (15), we can not get explicit form of b in terms of t and consequently all physical parameters cannot be determined in terms of t.

Case II:
Consider $D=0$ and $n=\frac{3}{2}$ in (14), we obtain

$$a=\left[\frac{6}{5}-\frac{1}{6}(t-t_0)^2\right]^{\frac{3}{2}}, \tag{16}$$

$$b=\frac{6}{5}-\frac{1}{6}(t-t_0)^2. \tag{17}$$

The model corresponding to solutions (16) and (17) is a contracting model which is not of much physical interest.

Case III:
Consider $D=0$ and $n=2$ in (14), we obtain

$$\int\left[\frac{1}{3}-\frac{1}{4}b^2\right]^{-\frac{1}{2}}db=\pm(t-t_0). \tag{18}$$

Solving (18), we get

$$a=\frac{4}{3}\sin^2\left(\frac{t-t_0}{2}\right), \tag{19}$$

$$b=\frac{2}{\sqrt{3}}\sin\left(\frac{t-t_0}{2}\right), \tag{20}$$

$$\phi=-\frac{9k}{16}\left[\cot^3\left(\frac{t-t_0}{2}\right)+\cos ec\left(\frac{t-t_0}{2}\right)\right].$$

Using (19) and (20), the line element (1) becomes

$$\begin{aligned} ds^2 &= -dt^2+\frac{16}{9}\sin^4\left(\frac{t-t_0}{2}\right)dx^2+\frac{4}{9}\sin^2\left(\frac{t-t_0}{2}\right)dy^2 \\ &\quad+\left[\frac{4}{9}\sin^2\left(\frac{t-t_0}{2}\right)\sin^2 y+\frac{16}{9}\sin^4\left(\frac{t-t_0}{2}\right)\cos^2 y\right]dz^2 \\ &\quad-\frac{32}{9}\sin^4\left(\frac{t-t_0}{2}\right)\cos y\,dx\,dz. \end{aligned} \tag{21}$$

From cases I, II and III, it should be noted that (14) can not be solved for any arbitrary values of n and D. We have worked out its solution for $D = 0$, $n = 2$ only. For $D = 0$, $n = 2$, we have obtained Bianchi type-IX cosmological model in presence of mass less scalar field.

3 Some Physical Properties of the Model

The model (21) represents an axially symmetric Bianchi type IX inflationary universe in general relativity when the scalar field is minimally coupled to the gravitational field in which the flat region of potential is constant which is generally associated with vacuum energy.

The model has no initial singularity at $t = 0$.

The physical and kinematical properties of the model (21) are as follows:

The spatial volume

$$= \sqrt{-g} = \frac{-256}{729} \sin^8\left(\frac{t - t_0}{2}\right) \sin^2 y.$$

Expansion scalar (θ)

$$\theta = 2\cot\left(\frac{t - t_0}{2}\right).$$

Shear scalar (σ^2)

$$\sigma^2 = 2\cot^2\left(\frac{t - t_0}{2}\right).$$

For this model (21), we have observed that the spatial volume tends to zero.

The scalar expansion and shear scalar tends to infinity as $t \to t_0$.

Also the deceleration parameter q is given by

$$q = -\frac{3}{\theta^2}\left[\theta_{;\alpha} U^\alpha + \frac{1}{3\theta^2}\right] = \frac{3}{4} \frac{1}{\cos^2(\frac{t-t_0}{2})} - 1.$$

We get negative value of deceleration parameter q as $t \to t_0$.

The negative value of the deceleration parameter q shows that model inflates.

We have

$$\lim_{T\to\infty}\left(\frac{\sigma}{\theta}\right) = \lim_{T\to\infty} \frac{\cot(\frac{t-t_0}{2})}{\sqrt{2}\cot(\frac{t-t_0}{2})} = \frac{1}{\sqrt{2}} \neq 0.$$

Therefore, the model does not approach isotropy for large values of $t = t_0$.

4 Conclusion

In this paper, we have obtained inflationary universe in the presence of mass less scalar field with a flat potential in general relativity. It is observed that the model is non-singular and does not approach isotropy for large values of t. The study of inflationary universe model has astrophysical significance in view of a recent interest in the classical scalar fields in general relativity and in alternative theories of gravitation.

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